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The integral homology of orientable Seifert manifolds

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Abstract

For any orientable Seifert manifold M, the integral homology group $H_1(M) = H_1(M;\mathbb{Z})$ is computed and explicit generators are found. This calculation gives a presentation for the p-torsion of $H_1(M)$ for any prime p. Since Seifert manifolds have dimension 3, $H_1(M)$ determines $H_*(M;A)$ and $H^*(M;A)$ as well, for any abelian group A. The complete details are given when $A = \mathbb{Z}, \mathbb{Z}/p^s$.

In order to calculate the partition functions of the Dijkgraaf–Witten topological quantum field theories it is necessary to compute the linking form of the underlying 3-manifold. In the case of the orientable Seifert manifolds it is possible to compute the linking form. The calculation of the linking form involves finding a presentation of the torsion of the first integral homology of the orientable Seifert manifolds, which is the main result of this paper.
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1. Introduction

The fundamental group $\pi_1(M)$ of any Seifert manifold M has been known since the publication of Seifert's original paper [19] in 1932. In principle the fundamental group determines $H_1(M) = (\pi_1(M))_{ab}$ (cf. [20]), but in practice the abelianization of $\pi_1(M)$

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is not a completely routine calculation. Define the p-component (cf. [18, Chapter 10]) of a finitely generated abelian group G, for any prime p, to be the quotient group G/H where H is the subgroup of all elements of G having (finite) order prime to p. The main result of this paper describes $H_1(M)$ in terms of its p-components and gives the generators and relations, for any orientable Seifert manifold M. This calculation was done by Seifert in the case when $H_1(M) = 0$ (in Seifert's terminology such a manifold is called a Poincaré fibred space). The necessary conditions given in Theorem 12 of [19] follow as a corollary of the results presented here. Furthermore, $H_1(M)$ completely determines $H_*(M; A)$ and $H^*(M; A)$ for any abelian group A.

In Section 2 of this paper, $H_1(M)$ is determined in terms of its p-components for the "genus 0" case, i.e., the case where the orbit surface of the Seifert manifold M is S^2. In Section 3 this is generalized to an arbitrary orientable Seifert manifold M, and is used to complete the calculation of $H_*(M; A)$ and $H^*(M; A)$ in the cases where $A = \mathbb{Z}$ or $\mathbb{Z}_{p^s}$. (When $s = 1$ these calculations are in complete agreement with previous work on $H^*(M; \mathbb{Z}/p)$ in [5,6].) Furthermore, Seifert's result for Poincaré fibred spaces is shown to be a corollary of these more general results.

Our motivation for finding a presentation for $H_1(M)$ comes from the study of topological quantum field theories (tqft's). There are virtually no examples where a tqft can be constructed from its associated quantum invariant (cf. [9]). However, a method for finding the partition functions of the Dijkgraaf–Witten tqft's (cf. [10]) is given in [1] and [2]. This method depends on the classification of linking pairings given in [15]. This in turn depends on determining the torsion of the first integral homology of these manifolds and this is explained in [2]. This method is now outlined.

The Dijkgraaf–Witten topological quantum field theories can be described as a topological gauge theory with finite gauge group (cf. [10]). In such a theory the only degree of freedom is the topology of the principal G-bundle over the 3-manifold M. For a discrete group all G bundles are necessarily flat and so are completely determined by homomorphisms of the fundamental group $\pi_1(M)$ into the group G, up to conjugation. There is an alternate description of the Dijkgraaf–Witten tqft's in terms of n-categories given by Freed in [11], and by Freed and Quinn in [12].

Such theories are discussed by Murakami et al. in [17]. In [17] the Dijkgraaf–Witten partition functions are described in terms of the Murakami–Ohtsuki–Okada invariants of orientable 3-manifolds. Furthermore, [17] describe a scheme to compute the partition function of the Dijkgraaf–Witten tqft's, however this scheme seems very difficult to implement in general.

On the other hand the [17] invariants are a special case of a new class of invariants, due to Deloup, now refered to as abelian WRT type invariants (cf. [7–9]) and these invariants can be described by an explicit formula that depends only on the linking form of the 3-manifold (cf. [8, Theorem 4]). The linking form of the 3-manifold, in turn, can be calculated from the mod p^s cup products and Bockstein maps of the 3-manifold, (cf. [21,22]). Thus, by using the description of the Dijkgraaf–Witten partition functions furnished by [17], it follows that they can be described in terms of the mod p^s cup products and Bockstein maps of the 3-manifold. A description of the relationship between the Dijkgraaf–Witten partition functions and the cohomology ring of the manifold is given in the introduction of [6] and in [1].

The calculation of the linking form of a 3-manifold depends on finding a presentation of the torsion of the first integral homology of the manifold. The object of this paper is to describe the torsion of the first integral homology when the 3-manifold is an orientable Seifert manifold, so that the Dijkgraaf–Witten invariants of such Seifert manifolds can be calculated in [1,2] using the results of this paper and those found in [3–6,8].

Using Seifert's notation, $M = (O, o, g \mid e\colon (a_1, b_1), \ldots, (a_n, b_n))$ denotes the orientable Seifert manifold with orientable orbit surface of genus $g \geqslant 0$ (i.e., the g-fold connected sum $T^2 \# \cdots \# T^2$), Euler number e, and n singular fibres whose twisting is described by the pairs of relatively prime integers (a_i, b_i). Similarly, $M = (O, n, g \mid e\colon (a_1, b_1), \ldots, (a_n, b_n))$ is the orientable Seifert manifold with orbit surface non-orientable of (non-orientable) genus $g \geqslant 1$ (i.e., the g-fold connected sum $\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$), with e, (a_i, b_i) as before. Some notation will now be described for the genus $g = 0$ case.

Given an orientable Seifert manifold $M \cong (O, o, 0 \mid e\colon (a_1, b_1), \ldots, (a_n, b_n))$,

$$H_1(M) \approx \Big\langle s_j, h \mid a_j s_j + b_j h = 0, \text{ for } j = 1, \ldots, n;\ \sum s_j - eh = 0 \Big\rangle.$$

Let $A = \prod a_j$, $A_j = A/a_j$ and $C = \sum b_j A_j$. Then $H_1(M)$ is a finite abelian group unless $Ae + C = 0$, in which case $H_1(M) \approx \mathbb{Z} \oplus \operatorname{Tor} H_1(M)$. For any integer m, let $\nu_p(m)$ denote the p-valuation of m, that is, the largest power of p that divides m. Note that $\nu_p(0) = \infty$. Order the Seifert invariants $a_1, a_2, \ldots, a_n$ so that $\nu_p(a_1) \leqslant \nu_p(a_2) \leqslant \cdots \leqslant \nu_p(a_n)$. Furthermore, let $F_p(G)$ denote the p-component of an abelian group G, and note that $\mathbb{Z}/p^0 = \{0\}$, $\mathbb{Z}/p^\infty = \mathbb{Z}$.

The two main theorems can now be stated. The first gives the p-component of $H_1(M)$ for the (Oo) case (the case where both M and its orbit surface are orientable). The second covers the (On) case (where M is orientable but its orbit surface is not) and gives $H_1(M)$ directly. For further details, including the explicit generators, in either case, see Sections 2 and 3.

Main Theorem 1. *For an orientable Seifert manifold $M = (O, o, g \mid e; (a_1, b_1), \ldots, (a_n, b_n))$, $g \geqslant 0$, the p-component of $H_1(M)$ is 0 if $\nu_p(a_{n-1}) = 0$, and otherwise (i.e., if $0 < \nu_p(a_{n-1}) \leqslant \nu_p(a_n)$) is:*

$$F_p H_1(M) \approx \mathbb{Z}^{2g} \oplus \mathbb{Z}/p^c \oplus \mathbb{Z}/p^{\nu_p(a_1)} \oplus \cdots \oplus \mathbb{Z}/p^{\nu_p(a_{n-2})},$$

where $c = \nu_p(Ae + C) - \nu_p(A) + \nu_p(a_{n-1}) + \nu_p(a_n)$.

Main Theorem 2. *Let $M = (O, n, g \mid e; (a_1, b_1), \ldots, (a_n, b_n))$, where $g > 0$.*

(1) *If $a_i \equiv 1 \pmod 2$ for all $1 \leqslant i \leqslant n$, then*

$$H_1(M) \approx \mathbb{Z}^{g-1} \oplus \begin{cases} (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/a_1 \oplus \cdots \oplus \mathbb{Z}/a_n & \text{if } \sum_{i=1}^n b_i + e \equiv 0 \pmod 2, \\ \mathbb{Z}/4 \oplus \mathbb{Z}/a_1 \oplus \cdots \oplus \mathbb{Z}/a_n & \text{if } \sum_{i=1}^n b_i + e \equiv 1 \pmod 2. \end{cases}$$

(2) *Suppose that $a_1 \equiv 0 \pmod 2$ (i.e., $\nu_2(a_1) > 0$). Define $q > 0$ by $\nu_2(a_1) = \nu_2(a_2) = \cdots = \nu_2(a_q) > \nu_2(a_{q+1})$. Then*

$$H_1(M) \approx \mathbb{Z}^{g-1} \oplus \begin{cases} \mathbb{Z}/4a_1 \oplus \mathbb{Z}/a_2 \oplus \cdots \oplus \mathbb{Z}/a_n & \text{if } q \equiv 1 \pmod 2, \\ \mathbb{Z}/2a_1 \oplus \mathbb{Z}/2a_2 \oplus \mathbb{Z}/a_3 \oplus \cdots \oplus \mathbb{Z}/a_n & \text{if } q \equiv 0 \pmod 2. \end{cases}$$

To illustrate Main Theorem 1 a typical example is given, complete with the explicit generators, at the end of Section 2. The MAPLE symbolic computation compiler was used to help formulate and check the results of this paper by using the ismith method which is described in [13].

2. The genus 0 case

Given an $m \times n$ matrix A with integer entries, it determines a linear transformation $\varphi_A : Z^m \to Z^n$ by:

$$\varphi_A\big([x_1, \dots, x_m]\big) = [x_1, \dots, x_m] \cdot A.$$

The product $A \cdot B$ of two matrices is defined if and only if the composition $\varphi_B \circ \varphi_A$ is; and in that case, $\varphi_{AB} = \varphi_B \circ \varphi_A$.

The transformation then determines a group $G_A = \operatorname{coker}(\varphi_A)$. Explicitly, G_A is the group obtained from the relations $r_1 = 0, \dots, r_m = 0$ on Z^n, where r_i is the ith row of the matrix A. The following lemma can be found in [16, p. 51] or [14, p. 299]. The proof is omitted.

Lemma 1 (Exact sequence of a composition). *For any groups and homomorphisms $\alpha : A \to B$, $\beta : B \to C$, there exists an exact sequence*

$$0 \to \ker(\alpha) \to \ker(\beta \circ \alpha) \xrightarrow{\alpha'} \ker(\beta) \xrightarrow{d} \operatorname{coker}(\alpha) \xrightarrow{\beta_*} \operatorname{coker}(\beta \circ \alpha) \xrightarrow{\pi} \operatorname{coker}(\beta) \to 0$$

where:

$$\alpha' = \alpha|_{\ker(\beta \circ \alpha)},$$
$$d(x) = x + \operatorname{Im}(\alpha),$$
$$\beta_*\big(x + \operatorname{Im}(\alpha)\big) = \beta(x) + \operatorname{Im}(\beta \circ \alpha),$$
$$\pi\big(x + \operatorname{Im}(\beta \circ \alpha)\big) = x + \operatorname{Im}(\beta).$$

Recall that Serre defines a $\mathcal{C}$-class of abelian groups to be a collection, $\mathcal{C}$, of abelian groups such that given a short exact sequence of abelian groups

$$0 \to A' \to A \to A'' \to 0$$

then $A \in \mathcal{C}$ if and only if both $A' \in \mathcal{C}$ and $A'' \in \mathcal{C}$ (cf. [18, Chapter 10] or [14, Chapter 10]). In particular let $\mathcal{C}_p$ denote the $\mathcal{C}$-class of all finite abelian groups having order relatively prime to p. A homomorphism $f : A \to B$ is said to be a $\mathcal{C}$-monomorphism if $\ker f \in \mathcal{C}$, a $\mathcal{C}$-epimorphism if $\operatorname{coker} f \in \mathcal{C}$, and a $\mathcal{C}$-isomorphism if both $\ker f \in \mathcal{C}$ and $\operatorname{coker} f \in \mathcal{C}$. Furthermore, let $F_p(G)$ denote the p-component of an abelian group G, that is, $F_p(G)$ is the quotient group obtained by factoring out the subgroup of all torsion elements of order prime to p. Lemma 1 from [18, p. 99] states that if $f : A \to B$ is a homomorphism of finitely generated abelian groups that is a $\mathcal{C}_p$-isomorphism, then f induces an isomorphism $F_p(A) \approx F_p(B)$. This lemma, which enables one to work “modulo $\mathcal{C}_p$”, will be frequently used in what follows, and will be referred to as the $\mathcal{C}_p$-isomorphism lemma.

Lemma 2 (Isomorphism of p-components).

(1) *Let M be an $m \times n$ matrix over $\mathbb{Z}$, and $A = \operatorname{diag}(q_1, \ldots, q_n)$ be an $n \times n$ diagonal matrix with each q_i relatively prime to p. Then there is an induced isomorphism $\varphi_{A*} : F_p(G_M) \to F_p(G_{MA})$ of the p-components.*

(2) *Similarly, for $B = \operatorname{diag}(q_1, \ldots, q_m)$ an $m \times m$ diagonal matrix, with each q_i relatively prime to p, there is an induced isomorphism $\pi : F_p(G_{BM}) \to F_p(G_M)$ of p-components.*

Proof. (1) Using Lemma 1 with $\beta = \varphi_A$, it is clear that β_* is a $\mathcal{C}_p$-isomorphism. The result now follows from the $\mathcal{C}_p$-isomorphism lemma.

(2) Similar to (1), with $\alpha = \varphi_B$. $\square$

Lemma 3. *Given integers p, e, and b_i $(1 \leqslant i \leqslant n)$, let $0 = k_0 \leqslant k_1 \leqslant \cdots \leqslant k_n$ be a sequence of integers such that $(p^{k_i}, b_i) = 1$ for all i. Furthermore, let u_i, v_i be integers such that $u_i p^{k_i - k_{i-1}} + v_i b_i = 1$. Now let $e_1, \ldots, e_{n+1}$ denote the standard basis of $\mathbb{Z}^{n+1}$ and define*:

$$\lambda_i = -S_i + p^{k_i - k_{i-1}} e_i + b_i U_1^{i-1} e_{n+1}, \quad \text{for } 1 \leqslant i \leqslant n-1,$$
$$\Lambda = -\sum_{j=1}^{n-1} U_{j+1}^{n-1} v_j e_j + U_1^{n-1} e_{n+1}, \qquad \tau = \sum_{j=1}^{n} e_j - e \cdot e_{n+1},$$

where

$$U_r^s = \begin{cases} 1 & \text{if } r > s, \\ \prod_{l=r}^{s} u_l & \text{if } r \leqslant s, \end{cases}$$
$$S_i = \begin{cases} 0 & \text{if } i < 2, \\ b_i \sum_{j=1}^{i-1} U_{j+1}^{i-1} v_j e_j & \text{if } i \geqslant 2. \end{cases}$$

Then τ, Λ, and the λ_i form a basis of $\mathbb{Z}^{n+1}$.

Before proceeding with the proof, observe that the generators given in Lemma 3 assume the following form:

$$\begin{aligned}
\lambda_1 &= p^{k_1} e_1 + b_1 e_{n+1}, \\
\lambda_2 &= -b_2 v_1 e_1 + p^{k_2 - k_1} e_2 + b_2 u_1 e_{n+1}, \\
\lambda_3 &= -b_3 u_2 v_1 e_1 - b_3 v_2 e_2 + p^{k_3 - k_2} e_3 + b_3 u_2 u_1 e_{n+1}, \\
\lambda_4 &= -b_4 u_3 u_2 v_1 e_1 - b_4 u_3 v_2 e_2 - b_4 v_3 e_3 + p^{k_4 - k_3} e_4 + b_4 u_3 u_2 u_1 e_{n+1}, \\
\lambda_5 &= -b_5 u_4 u_3 u_2 v_1 e_1 - b_5 u_4 u_3 v_2 e_2 - b_5 u_4 v_3 e_3 - b_5 v_4 e_4 \\
&\quad + p^{k_5 - k_4} e_5 + b_4 u_4 u_3 u_2 u_1 e_{n+1}, \\
&\ \vdots \\
\Lambda &= -v_1 u_2 u_3 \cdots u_{n-1} e_1 - v_2 u_3 \cdots u_{n-1} e_2 - \cdots \\
&\quad - v_{n-2} u_{n-1} e_{n-2} - v_{n-1} e_{n-1} + u_1 u_2 u_3 \cdots u_{n-1} e_{n+1}, \\
\tau &= e_1 + e_2 + e_3 + \cdots + e_n - e \cdot e_{n+1}.
\end{aligned}$$

Proof. It follows immediately from the definition of λ_i, Λ, and τ that

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \\ \Lambda \\ \tau \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \\ e_{n+1} \end{pmatrix},$$

where the (i, j) entry of the matrix A is defined by:

$$a_{i,j} = \begin{cases} 0 & \text{if } i < j < n+1, \\ p^{k_i - k_{i-1}} & \text{if } i = j < n, \\ b_i U_1^{i-1} & \text{if } i < n,\ j = n+1, \\ -b_i v_j U_{j+1}^{i-1} & \text{if } j < i < n, \\ -v_j U_{j+1}^{n-1} & \text{if } i = n,\ j < n, \\ 0 & \text{if } i = j = n, \\ U_1^{n-1} & \text{if } i = n,\ j = n+1, \\ 1 & \text{if } i = n+1,\ j < n+1, \\ -e & \text{if } i = j = n+1. \end{cases}$$

To complete the proof it suffices to show that $|\det(A)| = 1$.

Proceed by descending induction on q, from $q = n-1$ to 1. For each value of q, perform the following operations on A:

(1) Multiply row q by u_q;
(2) Subtract b_q times row n from row q;
(3) Add v_q times row q to row n;
(4) Divide row n by u_q.

These operations modify row n and row q, and performing all four leaves $\det(A)$ unchanged. To follow the effect of these row operations more easily see Example 1 below, which gives A in the case $n = 4$.

Assume, as the induction hypothesis, that row n, before operation q takes place, is of the form:

$$a'_{n,j} = \begin{cases} 0 & \text{if } l < j < n+1, \\ -v_j U_{j+1}^{q} & \text{if } j \leqslant q, \\ U_1^{q} & \text{if } j = n+1. \end{cases}$$

The initial case, with $q = n-1$, is obviously true.

The first two steps change row q to the form

$$a'_{q,j} = \begin{cases} 0 & \text{if } j \neq q, \\ 1 & \text{if } j = q. \end{cases}$$

The second two steps change row n to the form

$$a''_{n,j} = \begin{cases} 0 & \text{if } q-1 < j < n+1, \\ -v_j U_{j+1}^{q-1} & \text{if } j \leqslant q-1, \\ U_1^{q-1} & \text{if } j = n+1 \end{cases}$$

which is the form row n has before operation $(q-1)$ takes place.

The preceding $n-1$ operations yield a new matrix B with the same determinant as A, having the form

$$b_{i,j} = \begin{cases} 1 & \text{if } i < n,\ i = j, \\ 0 & \text{if } i < n,\ i \neq j, \\ 1 & \text{if } i = n,\ j = n+1, \\ 0 & \text{if } i = n,\ j \neq n+1, \\ 1 & \text{if } i = n+1,\ j < n+1, \\ -e & \text{if } i = n+1,\ j = n+1. \end{cases}$$

Thus the determinant must be either 1 or -1. □

Example 1. For $n = 4$,

$$A = \begin{pmatrix} p^{k_1} & 0 & 0 & 0 & b_1 \\ -b_2v_1 & p^{k_2} & 0 & 0 & b_2U_1^1 \\ -b_3v_1U_2^2 & -b_3v_2 & p^{k_3} & 0 & b_3U_1^2 \\ -v_1U_2^3 & -v_2U_3^3 & -v_3 & 0 & U_1^3 \\ 1 & 1 & 1 & 1 & -e \end{pmatrix}$$
$$= \begin{pmatrix} p^{k_1} & 0 & 0 & 0 & b_1 \\ -b_2v_1 & p^{k_2} & 0 & 0 & b_2u_1 \\ -b_3v_1u_2 & -b_3v_2 & p^{k_3} & 0 & b_3u_1u_2 \\ -v_1u_2u_3 & -v_2u_3 & -v_3 & 0 & u_1u_2u_3 \\ 1 & 1 & 1 & 1 & -e \end{pmatrix}.$$

Lemma 4. *Assume that the hypothesis given in Lemma* 3 *is satisfied. Let N denote the subgroup of $\mathbb{Z}^{n+1}$ generated by τ and all elements of the form $p^{k_i}e_i + b_ie_{n+1}$, for $1 \leqslant i \leqslant n$. Let N' be the subgroup of $\mathbb{Z}^{n+1}$ generated by τ, $p^{k_{i-1}}\lambda_i$ for all $1 \leqslant i \leqslant n-1$, and $\mu\Lambda$, where*

$$\mu = p^{k_n+k_{n-1}}\left(e + \sum_{l=1}^{n} \frac{b_l}{p^{k_l}}\right) \in \mathbb{Z}.$$

Then $N = N'$.

Proof. Consider the elements

$$\tau,$$
$$\lambda_1 = p^{k_1}e_1 + b_1e_{n+1},$$

$$\begin{aligned}
p^{k_{i-1}}\lambda_i &= p^{k_i}e_i + b_i p^{k_{i-1}}U_1^{i-1}e_{n+1} - p^{k_{i-1}}S_i \\
&= \left(p^{k_i}e_i + b_i e_{n+1}\right) - \sum_{l=1}^{i-1}\left[p^{k_{i-1}-k_l}b_i v_l U_{l+1}^{i-1}\left(p^{k_l}e_l + b_l e_{n+1}\right)\right],
\end{aligned}$$

for $2 \leqslant i \leqslant n-1$,

$$\begin{aligned}
\mu\Lambda &= p^{k_n+k_{n-1}}\left(e + \sum_{l=1}^{n}\frac{b_l}{p^{k_l}}\right)\left[-\sum_{j=1}^{n-1}U_{j+1}^{n-1}v_j e_j + U_1^{n-1}e_{n+1}\right] \\
&= p^{k_n}\tau - \left(p^{k_n}e_n + b_n e_{n+1}\right) + \sum_{l=1}^{n-1}\left(-p^{k_n-k_l} + \mu v_l p^{-k_l}U_{l+1}^{n-1}\right)\left(p^{k_l}e_l + b_l e_{n+1}\right).
\end{aligned}$$

(The expressions for $p^{k_i-1}\lambda_i$ and $\mu\Lambda$ can be deduced from $p^{k_i-1}U_1^{i-1} = 1 - \sum_{l=1}^{i-1} p^{k_{i-1}-k_l}v_l b_l U_{l+1}^{i-1}$.) These elements can be written as a column matrix as follows:

$$\begin{pmatrix} \tau \\ \lambda_1 \\ p^{k_1}\lambda_2 \\ p^{k_2}\lambda_3 \\ \vdots \\ p^{k_{n-2}}\lambda_{n-1} \\ \mu\Lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & * & 1 & 0 & & 0 & 0 \\ 0 & * & * & 1 & & 0 & 0 \\ & \vdots & & & \ddots & & \vdots \\ 0 & * & * & * & & 1 & 0 \\ * & * & * & * & \cdots & * & -1 \end{pmatrix} \begin{pmatrix} \tau \\ p^{k_1}e_1 + b_1 e_{n+1} \\ p^{k_2}e_2 + b_2 e_{n+1} \\ \vdots \\ p^{k_n}e_n + b_n e_{n+1} \end{pmatrix}.$$

The lower triangular matrix above obviously has determinant -1, so is invertible over $\mathbb{Z}$, and hence $N = N'$. $\square$

Corollary 1. *If M is the matrix*:

$$\begin{pmatrix} 1 & 1 & 1 & & 1 & -e \\ p^{k_1} & 0 & 0 & \cdots & 0 & b_1 \\ 0 & p^{k_2} & 0 & & 0 & b_2 \\ 0 & 0 & p^{k_3} & & 0 & b_3 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & p^{k_n} & b_n \end{pmatrix},$$

where $0 = k_0 \leqslant k_1 \leqslant \cdots \leqslant k_n$ *and* $(b_i, p) = 1$ *for all* i, *then*

$$G_M = \left\langle \lambda_i, \Lambda \mid p^{k_{i-1}}\lambda_i = 0,\ 2 \leqslant i \leqslant n-1,\ \mu\Lambda = 0 \right\rangle.$$

Proof. It suffices to note that in G_M, $\tau = 0$ and $\lambda_1 = 0$. $\square$

Lemma 5. *Let a_i, b_i denote relatively prime integers, for all i. Suppose that $a_i = x_i p^{k_i}$, where $(x_i, p) = 1$, and $0 \leqslant k_1 \leqslant \cdots \leqslant k_n$. If*

$$M = \begin{pmatrix} 1 & 1 & 1 & & 1 & -e \\ x_1 p^{k_1} & 0 & 0 & \cdots & 0 & b_1 \\ 0 & x_2 p^{k_2} & 0 & & 0 & b_2 \\ 0 & 0 & x_3 p^{k_3} & & 0 & b_3 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & x_n p^{k_n} & b_n \end{pmatrix},$$

then there is an isomorphism $f : F_p(G_M) \to F_p(G_{M'})$, where

$$M' = \begin{pmatrix} 1 & 1 & 1 & & 1 & -e \prod x_j \\ p^{k_1} & 0 & 0 & \cdots & 0 & b_1 \frac{\prod x_j}{x_1} \\ 0 & p^{k_2} & 0 & & 0 & b_2 \frac{\prod x_j}{x_2} \\ 0 & 0 & p^{k_3} & & 0 & b_3 \frac{\prod x_j}{x_3} \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & p^{k_n} & b_n \frac{\prod x_j}{x_n} \end{pmatrix}$$

and $f([a_1, \ldots, a_{n+1}]) = [a_1, \ldots, a_n, a_{n+1} \prod x_i]$.

Proof. Let $A = \mathrm{diag}(1, 1, \ldots, 1, \prod x_j)$ and $B = \mathrm{diag}(1, x_1, \ldots, x_n)$ be $(n+1) \times (n+1)$-diagonal matrices. Then $MA = BM'$.

Since $(x_j, p) = 1$ for all j, $(\prod x_j, p) = 1$. By the isomorphism of p-components (Lemma 2) there is an isomorphism

$$F_p(G_M) \xrightarrow{\varphi_{A*}} F_p(G_{MA}) = F_p(G_{BM'}) \xrightarrow{\pi_1} F_p(G_{M'}),$$

where the composite map $f = \pi_1 \circ \varphi_{A*} : F_p(G_M) \to F_p(G_{M'})$ is an isomorphism that clearly satisfies the property $f([a_1, \ldots, a_{n+1}]) = [a_1, \ldots, a_n, a_{n+1} \prod x_i]$. $\square$

Lemma 6. *Choose integers u_i, v_i, such that $u_i p^{k_i - k_{i-1}} + v_i b_i \frac{\prod x_j}{x_i} = 1$. Then $F_p(G_M)$ is given by the generators*:

$$\gamma_i = -T_i + x_i p^{k_i - k_{i-1}} e_i + b_i U_1^{i-1} e_{n+1}, \quad \text{for } 2 \leqslant i \leqslant n-1,$$

$$\Gamma = \frac{\omega}{p^{\nu_p(\omega)}} \left[-\left(\prod x_l\right) \sum_{j=1}^{n-1} U_{j+1}^{n-1} v_j e_j + U_1^{n-1} e_{n+1} \right],$$

and relations:

$$p^{k_{i-1}} \gamma_i = 0,$$
$$p^{\nu_p(\omega)} \Gamma = 0,$$

where

$$\omega = \Bigl(\prod x_l\Bigr) p^{k_n+k_{n-1}} \left(e + \sum_{l=1}^{n} \frac{b_l}{a_l}\right),$$
$$U_r^s = \begin{cases} 1 & \text{if } r > s, \\ \prod_{l=r}^{s} u_l, & \text{if } r \leqslant s, \end{cases}$$
$$T_i = \begin{cases} 0 & \text{if } i < 2, \\ b_i\bigl(\prod x_l\bigr) \sum_{j=1}^{i-1} U_{j+1}^{i-1} v_j e_j & \text{if } i \geqslant 2. \end{cases}$$

Proof. It follows from Corollary 1 that $G_{M'}$ is generated by the elements

$$\begin{aligned} \lambda_i \;&=\; -\left(\frac{b_i \prod x_l}{x_i}\right) \sum_{j=1}^{i-1} \bigl[U_{j+1}^{i-1} v_j e_j\bigr] + p^{k_i - k_{i-1}} e_i \\ &\quad + b_i\left(\frac{\prod x_l}{x_i}\right) U_1^{i-1} e_{n+1}, \quad \text{for } 1 \leqslant i \leqslant n-1, \\ \Lambda \;&=\; -\sum_{j=1}^{n-1} \bigl[U_{j+1}^{n-1} v_j e_j\bigr] + U_1^{n-1} e_{n+1}, \end{aligned}$$

which satisfy the relations

$$p^{k_{i-1}} \lambda_i = 0, \quad \text{for } 2 \leqslant i \leqslant n-1,$$
$$\omega \Lambda = 0,$$

where ω is defined above.

Thus, the p-component of $G_{M'}$ is generated by the elements λ_i and $\frac{\omega}{p^{\nu_p(\omega)}}\Lambda$, and these generators satisfy the relations $p^{k_{i-1}}\lambda_i = 0$, $p^{\nu_p(\omega)}\frac{\omega}{p^{\nu_p(\omega)}}\Lambda = 0$. Since $(x_i, p) = 1$, $x_i\lambda_i$ can replace λ_i as a generator for all i, and similarly $(\prod x_j)(\frac{\omega}{p^{k_\omega}})\Lambda$ can replace Λ. So, an alternate presentation of $F_p(G_{M'})$ is given by the generators $x_i\lambda_i$ and $(\prod x_j)(\frac{\omega}{p^{\nu_p(\omega)}})\Lambda$, and relations $p^{k_{i-1}}(x_i\lambda_i) = 0$ and $p^{\nu_p(\omega)}(\prod x_j)(\frac{\omega}{p^{\nu_p(\omega)}})\Lambda = 0$.

However, if $f : F_p(G_M) \to F_p(G_{M'})$ is the isomorphism defined in Lemma 5, then in $F_p(G_{M'})$:

$$[x_i\lambda_i] = f\bigl([\gamma_i]\bigr),$$
$$\Bigl(\prod x_j\Bigr)\left(\frac{\omega}{p^{\nu_p(\omega)}}\right)\Lambda = f\bigl([\Gamma]\bigr).$$

Since f is an isomorphism, it follows that $F_p(G_M)$ is given by the generators γ_i, Γ and relations $p^{k_{i-1}}\gamma_i = 0$, $p^{\nu_p(\omega)}\Gamma = 0$. □

The algebraic considerations given above will now be applied to finding $H_1(M)$, where $M = (O, o, 0 \mid e\colon (a_1, b_1), \ldots, (a_n, b_n))$ is an orientable Seifert manifold whose orbit surface has genus 0, i.e., the orbit surface of M is S^2. In fact, by applying Lemmas 5 and 6, this calculation is now trivial, but it will be convenient to fix (and recall) some notation. Each (a_i, b_i) is a pair of relatively prime integers, p a fixed prime $a_i = x_i p^{\nu_p(a_i)}$ (x_i relatively prime to p), $A = \prod_{i=1}^{n} a_i$, $A_j = A/a_j$, $C = \sum_{j=1}^{n} b_j A_j$. Assume that

$0 \leqslant \nu_p(a_1) \leqslant \cdots \leqslant \nu_p(a_n)$, and for each i choose $u_i, v_i \in \mathbb{Z}$ such that

$$u_i p^{\nu_p(a_i)-\nu_p(a_{i-1})} + v_i b_i \left(\prod_{i \neq j} x_j \right) = 1.$$

Finally, define

$$\omega = \frac{(Ae + C)}{\prod_{i=1}^{n-2} p^{\nu_p(a_i)}} = \left(\prod_{i=1}^{n} x_i \right) p^{\nu_p(a_{n-1})+\nu_p(a_n)} \left(e + \sum_{i=1}^{n} \frac{b_i}{a_i} \right),$$

and recall that $H_1(M)$ has a natural presentation: $H_1(M) \approx \langle s_1, \ldots, s_n, h \mid a_i s_i + b_i h = 0, 1 \leqslant i \leqslant n, \sum s_i - eh = 0 \rangle$, where the s_i and H are the abelianizations of standard generators of $\pi_1(M)$ (cf. [19]). The following theorem gives $F_p H_1(M)$, and its generators, as linear combinations of the standard generators mentioned above.

Theorem 1 (Genus 0 case). *For a Seifert manifold $M = (O, o, 0 \mid e; (a_1, b_1), \ldots, (a_n, b_n))$ the p-component of $H_1(M)$ is*:

$$F_p\big(H_1(M)\big) \approx \mathbb{Z}/p^c \oplus \mathbb{Z}/p^{\nu_p(a_1)} \oplus \cdots \oplus \mathbb{Z}/p^{\nu_p(a_{n-2})},$$

where $c = \nu_p(Ae + C) - \nu_p(A) + \nu_p(a_{n-1}) + \nu_p(a_n)$.

Furthermore a presentation for the p-component is:

$$F_p\big(H_1(M; \mathbb{Z})\big) = \big\langle \Gamma, \gamma_2, \ldots, \gamma_{n-1} \mid p^{\nu_p(a_{i-1})} \gamma_i = 0,\ p^c \Gamma = 0,\ \textit{for } i = 2, \ldots, n-1 \big\rangle,$$

where

$$\gamma_i = -T_i + x_i p^{\nu_p(a_i)-\nu_p(a_{i-1})} s_i + b_i U_1^{i-1} h, \quad \textit{for } 2 \leqslant i \leqslant n-1,$$

$$\Gamma = -\frac{\omega}{p^{\nu_p(\omega)}} \left[\left(\prod x_l \right) \sum_{j=1}^{n-1} U_{j+1}^{n-1} v_j s_j + U_1^{n-1} h \right],$$

and

$$U_r^s = \begin{cases} 1 & \textit{if } r > s, \\ \prod_{l=r}^{s} u_l & \textit{if } r \leqslant s, \end{cases}$$

$$T_i = \begin{cases} 0 & \textit{if } i < 2, \\ b_i \left(\prod x_l \right) \sum_{j=1}^{i-1} U_{j+1}^{i-1} v_j s_j & \textit{if } i \geqslant 2. \end{cases}$$

Proof. Making the identifications $e_1 = s_1, \ldots, e_n = s_n, e_{n+1} = h$, the theorem follows at once from Lemmas 5 and 6, since the matrix in Lemma 5 represents precisely the same relations given in the presentation of $H_1(M)$. It only remains to prove that $\nu_p(\omega) = c$. But $\nu_p(\omega) = \nu_p(Ae + C) - \sum_{i=1}^{n-2} \nu_p(a_i) = \nu_p(Ae + C) - \nu_p(A) + \nu_p(a_{n-1}) + \nu_p(a_n) = c$. $\square$

Remark 1. It should be noted that the generators given in Theorem 1 for the p-components of $H_1(M)$ are not necessarily p-torsion elements in $H_1(M)$ itself. However, an argument similar to that given Lemma 4 can be used to show the these generators do in fact have the same orders in $H_1(M)$ (i.e., $p^{\nu_p(a_{i-1})}$ for γ_i and p^c for Γ) and so, all together, form a basis of $H_1(M)$. The factor $\frac{\omega}{p^{\nu_p(\omega)}}$ in the expression for Γ, being relatively prime to p, has no effect on the p-component $F_p(H_1(M))$ but is crucial to Γ having order precisely p^c in $H_1(M)$.

Example 2. Consider the Seifert manifold $M = (O, o, 0 \mid 1; (2,3), (12,1), (8,7), (40,33))$. Apply Theorem 1 to calculate the p-components of $H_1(M)$ and their generators. First of all observe that $A = \prod a_j = (2)(12)(8)(40) = (2^9)(3)(5)$, and $Ae + C = (2^7)(257)$.

For $p = 2$, observe that $\nu_2(a_1) = \nu_2(2) = 1$, $\nu_2(a_2) = \nu_2(12) = 2$ and $c = \nu_2(Ae + C) - \nu_2(A) + \nu_2(a_3) + \nu_p(a_4) = 4$. Thus

$$F_2\big(H_1(M)\big) = \mathbb{Z}/2^4 \oplus \mathbb{Z}/2^1 \oplus \mathbb{Z}/2^2 = \mathbb{Z}/16 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4.$$

The generators of $F_2(H_1(M))$ are:

$$\gamma_2 = -15s_1 + 6s_2 - 22h,$$
$$\gamma_3 = 210s_1 - 105s_2 + 2s_3 + 308h,$$
$$\Gamma = -400920s_1 + 200460s_2 - 3855s_3 + 588016h.$$

For $p = 3$, $F_3(H_1(M)) = \mathbb{Z}/3^0 \oplus \mathbb{Z}/3^0 \oplus \mathbb{Z}/3^0 = 0$. Similarly when $p = 5$, $F_5(H_1(M)) = 0$.

Finally, when $p = 257$, $\nu_{257}(2) = \nu_{257}(12) = \nu_{257}(8) = \nu_{257}(40) = 0$ and $c = 1$. Whence,

$$F_{257}\big(H_1(M)\big) = \mathbb{Z}/257.$$

In this case there is only one generator which is:

$$\Gamma' = -4220624240640s_1 + 6605045760s_2 - 983040s_3 - 6330386800512h.$$

3. Generalization to the genus $g > 0$ case

The results of Section 2 pertaining to Seifert manifolds with orbit surface S^2 will be generalized in this section to arbitrary orientable Seifert manifolds $M = (O, o, g \mid e; (a_1, b_1), \ldots, (a_n, b_n))$, for $g \geqslant 0$ or $M = (O, n, g \mid e; (a_1, b_1), \ldots, (a_n, b_n))$, for $g \geqslant 1$.

In the (Oo) case recall that

$$\begin{aligned}\pi_1(M) = \big\langle & s_1, \ldots, s_n, v_1, w_1, \ldots, v_g, w_g, h \mid \\ & [s_j, h], s_j^{a_j} h^{b_j}, [v_j, h], [w_j, h], s_1 \cdots s_n [v_1, w_1] \cdots [v_g, w_g] h^{-e} \big\rangle.\end{aligned}$$

The next result generalizes Theorem 1 in Section 2, and gives the generators of $F_p(H_1(M))$ in terms of the (abelianized) standard generators s_i, v_k, w_k of $\pi_1(M)$. (This theorem includes the result given in Main Theorem 1.)

Theorem 2. *Let $M = (O, o, g \mid e; (a_1, b_1), \ldots, (a_n, b_n))$, with $0 \leqslant \nu_p(a_1) \leqslant \cdots \leqslant \nu_p(a_n)$. Then*

$$F_p\big(H_1(M)\big) \approx \mathbb{Z}^{2g} \oplus \mathbb{Z}/p^c \oplus \mathbb{Z}/p^{\nu_p(a_1)} \oplus \cdots \oplus \mathbb{Z}/p^{\nu_p(a_{n-2})}.$$

Furthermore a presentation for the p-component is:

$$F_p\big(H_1(M;\mathbb{Z})\big) = \big\langle v_1, w_1, \ldots, v_g, w_g, \Gamma, \gamma_2, \ldots, \gamma_{n-1} \mid p^{\nu_p(a_{i-1})}\gamma_i = 0,\ p^c\Gamma = 0,\ \textit{for } i = 2, \ldots, n-1\big\rangle,$$

where c, Γ, γ_i are defined in Section 2.

Proof. The generators $v_1, w_1, \ldots, v_g, w_g$, vanish from the relations in $\pi_1(M)_{\mathrm{ab}}$ and hence generate $2g$ free cyclic summands in $H_1(M) = \pi_1(M)_{\mathrm{ab}}$. The remaining generators are obtained from Theorem 1 in Section 2. □

Turning to the case where $M = (O, n, g \mid e; (a_1, b_1), \ldots, (a_n, b_n))$, $g > 0$, recall that

$$\pi_1(M) = \big\langle s_1, \ldots, s_n, v_1, \ldots, v_g, h \mid [s_j, h], s_j^{a_j} h^{b_j}, v_k h v_k^{-1} h, s_1 \cdots s_n v_1^2 \cdots v_g^2 h^{-e}\big\rangle.$$

Abelianizing the fundamental group gives

$$H_1(M) = \big\langle s_1, \ldots, s_n, v_1, \ldots, v_g, h \mid a_i s_i + b_i h = 0,\ 2h = 0,\ s + 2v - eh = 0\big\rangle, \quad (1)$$

where $s = \sum_{i=1}^n s_i$ and $v = \sum_{k=1}^g v_k$. The fact that $2h = 0$ here simplifies the (On) case. In fact $H_1(M)$ can be expressed as an internal direct sum without resorting to the calculation of the p-components as in the (Oo) case.

As before let $A = \prod a_i$ and assume without loss of generality that $0 \leqslant \nu_2(a_1) \leqslant \cdots \leqslant \nu_2(a_n)$. Now write $a_i = 2^{\nu_2(a_i)} x_i$ where $x_i \equiv 1 \pmod 2$. The next theorem includes the result of Main Theorem 2, stated in the Introduction.

Theorem 3. *Let $M = (O, n, g \mid e; (a_1, b_1), \ldots, (a_n, b_n))$, where $g > 0$.*

(1) *If $a_i \equiv 1 \pmod 2$ for all $1 \leqslant i \leqslant n$, then*

$$H_1(M) \approx \mathbb{Z}^{g-1} \oplus \begin{cases} (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/a_1 \oplus \cdots \oplus \mathbb{Z}/a_n & \textit{if } \sum_{i=1}^n b_i + e \equiv 0 \pmod 2, \\ \mathbb{Z}/4 \oplus \mathbb{Z}/a_1 \oplus \cdots \oplus \mathbb{Z}/a_n & \textit{if } \sum_{i=1}^n b_i + e \equiv 1 \pmod 2. \end{cases}$$

In the first case, the generators are $v_1, \ldots, v_{g-1}, h, v', s_1', \ldots, s_n'$ respectively, where $s_i' = s_i + b_i h$, and $v' = v + \frac{(A+1)}{2} s$, while in the second case the generators are $v_1, \ldots, v_{g-1}, v', s_1', \ldots, s_n'$, respectively.

(2) *Suppose that $a_1 \equiv 0 \pmod 2$ (i.e., $\nu_2(a_1) > 0$). Set $s_i' = s_i + 2^{\nu_2(a_1) - \nu_2(a_i)} x_1 b_i s_1$ for $2 \leqslant i \leqslant n$ and define $q > 0$ by $\nu_2(a_1) = \nu_2(a_2) = \cdots = \nu_2(a_q) > \nu_2(a_{q+1})$.*

(a) *If $q \equiv 1 \pmod 2$, define*

$$a = \frac{x_1}{2}\left(-\sum_{i=2}^{q} b_i + 2^{\nu_2(a_1)} e - \sum_{i=q+1}^{n} 2^{\nu_2(a_1) - \nu_2(a_i)} b_i\right),$$

$$v' = v + a s_1 - \sum_{i=2}^{n} \frac{x_i - 1}{2} s_i'.$$

Then

$$H_1(M) \approx \mathbb{Z}^{g-1} \oplus \mathbb{Z}/4a_1 \oplus \mathbb{Z}/a_2 \oplus \cdots \oplus \mathbb{Z}/a_n,$$

has respective generators $v_1, \ldots, v_{g-1}, v', s_2', \ldots, s_n'$.

(b) *If* $q \equiv 0 \pmod 2$, *set*

$$v' = v + \left(\frac{x_1 - 1}{2}\right)s_1 + \sum_{i=3}^{q} \frac{x_i - 1}{2} s_i'.$$

Then

$$H_1(M) \approx \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2a_1 \oplus \mathbb{Z}/2a_2 \oplus \mathbb{Z}/a_3 \oplus \cdots \oplus \mathbb{Z}/a_n,$$

has respective generators $v_1, \ldots, v_{g-1}, s_1, v', s_3', \ldots, s_n'$.

Proof. Note from (1) that in all cases $h = \pm h = xh$, for $x \equiv 1 \pmod 2$, and

$$b_i h = \begin{cases} 0, & \text{if } b_i \equiv 0 \pmod 2, \\ h, & \text{if } b_i \equiv 1 \pmod 2. \end{cases}$$

Also observe that in each case the relations in $H_1(M)$ only involve $v = \sum_{k=1}^{g} v_k$, hence $v_1, \ldots, v_{g-1}$ generate $\mathbb{Z}^{g-1}$. It remains to prove that the subgroup G of $H_1(M)$ defined by

$$G = \langle s_1, s_2, \ldots, s_n, v, h \mid a_i s_i + b_i h = 0,\ 2h = 0,\ s + 2v + eh = 0 \rangle,$$

gives the torsion subgroup of $H_1(M)$, and has the stated generators in each case.

In case (1), the transition matrix of the relation

$$\begin{pmatrix} s_1' \\ s_2' \\ \vdots \\ s_n' \\ v' \\ h \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & b_1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & b_2 \\ & \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & b_n \\ a & a & a & \cdots & a & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \\ v \\ h \end{pmatrix}$$

has determinant $+1$, so gives an admissible change of generators. Now $a_i s_i' = a_i(s_i + b_i h) = a_i s_i + b_i(a_i h) = a_i s_i + b_i h = 0$, while

$$\begin{aligned} 2v' &= 2v + (A+1)s = eh - s + As + s = eh + \sum_{i=1}^{n} As_i \\ &= eh + \sum_{i=1}^{n}\left(\frac{A}{a_i}\right)a_i s_i = eh + \sum_{i=1}^{n}\left(\frac{A}{a_i}\right)b_i h = eh + \sum_{i=1}^{n} b_i h, \end{aligned}$$

$$\left(e + \sum_{i=1}^{n} b_i\right)h = \begin{cases} 0, & \text{if } \sum_{i=1}^{n} b_i + e \equiv 0 \pmod 2, \\ h, & \text{if } \sum_{i=1}^{n} b_i + e \equiv 1 \pmod 2. \end{cases}$$

It follows that $G = \mathrm{Tor}(H_1(M))$ and hence $H_1(M)$ has the exact generators stated.

For 2(a) and 2(b) first note that $a_i \equiv 0 \pmod 2$ implies that $b_i \equiv 1 \pmod 2$. In particular $b_1, \ldots, b_q \equiv 1 \pmod 2$, and therefore $a_1 s_1 = h$. So G is generated by $\{s_1, \ldots, s_n, v\}$. The change of generators from $\{s_1, \ldots, s_n, v\}$ to $\{s_1, s_2', \ldots, s_n', v'\}$, in 2(a) or 2(b), is clearly unimodular. Furthermore, in both of these cases,

$$\begin{aligned} a_i s_i' &= a_i s_i + 2^{\nu_2(a_1)-\nu_2(a_i)} a_i x_1 b_i s_1 = b_i h + 2^{\nu_2(a_1)-\nu_2(a_i)} 2^{\nu_2(a_i)} x_i x_1 b_i s_1 \\ &= b_i h + x_i b_i h = b_i h + b_i h = 0. \end{aligned}$$

The remaining relation $s + 2v + eh = 0$ in either 2(a) or 2(b) reduces to

$$\left[1 + x_1\left(-\sum_{i=2}^{q} b_i + 2^{\nu_2(a_1)} e - \sum_{i=q+1}^{n} 2^{\nu_2(a_1)-\nu_2(a_i)} b_i\right)\right] s_1 + s_2' + \cdots + s_n' + 2v = 0. \tag{2}$$

In case 2(a) with q odd, the number a defined in the statement of 2(a) is integral. Since all x_i are odd, $(x_i - 1)/2$ is an integer and thus the coefficients in the expression for v' are integral. In terms of v', (2) transforms to

$$2v' + s_1 + \sum_{i=2}^{n} x_i s_i' = 0. \tag{3}$$

This shows that G is generated by $s_2', \ldots, s_n', v'$. Furthermore, multiplying (3) by $x_1 2^{\nu_2(a_1)} = a_1$ gives $x_1 2^{\nu_2(a_1)+1} v' + a_1 s_1 = 0$, i.e., $x_1 2^{\nu_2(a_1)+1} v' = h$ and $0 = x_1 2^{\nu_2(a_1)+2} v' = 4a_1 v'$.

Finally, for case 2(b) with q even (so $q \geqslant 2$ and $\nu_2(a_1) = \nu_2(a_2)$), converting (2) to v' gives

$$2v' + (2a + x_1 - 1)s_1 + s_2' + \sum_{i=3}^{n} x_i s_i' = 0, \tag{4}$$

where

$$2a = 1 + x_1\left(-\sum_{i=2}^{q} b_i + 2^{\nu_2(a_1)} e - \sum_{i=q+1}^{n} 2^{\nu_2(a_1)-\nu_2(a_i)} b_i\right) = 1 + x_1(2b+1)$$

for suitable $b \in \mathbb{Z}$. It follows that (4) can be written

$$2v' + 2x_1(b+1)s_1 + s_2' + \sum_{i=3}^{n} x_i s_i' = 0. \tag{5}$$

In particular G is generated by $v', s_1, s_3', \ldots, s_n'$, and the relation $2a_2 v' = 2^{\nu_2(a_1)+1} x_2 v' = 0$ follows from (5). $\square$

This completes the proofs of Theorems 2 and 3 (and includes the results stated in Main Theorems 1 and 2). The remaining results, Corollaries 2 and 3, are essentially simple consequences of these theorems, starting with a result that determines the necessary conditions for a Seifert manifold to be a homology sphere.

Corollary 2. *Given a Seifert manifold M, if $H_1(M) = 0$ then $M = (O, o, 0 \mid e; (a_1, b_1), \ldots, (a_n, b_n))$ and the a_j are pairwise relatively prime.*

Proof. If $H_1(M) = 0$ then M is orientable (since it follows that $H^1(M; \mathbb{Z}/2) = 0$ and hence the first Stiefel–Whitney class $w_1(M) = 0$). If $g > 0$ Theorem 2 shows that $H_1(M)$

is infinite in the (Oo) case, while Theorem 3 shows that $H_1(M)$ has non-trivial 2-torsion in the (On) case. Thus $g = 0$. (Which implies that $M = (O, o, 0)$.) Now for any prime p with $\nu_p(a_{n-1}) > 0$, it follows from Theorem 1 that $c = \nu_p(A) - \nu_p(a_n) - \nu_p(A) + \nu_p(a_{n-1}) + \nu_p(a_n) = \nu_p(a_{n-1}) > 0$. This means that $H_1(M)$ must have a non-trivial p-component. Hence $\nu_p(a_{n-1}) = 0$ for all p, which is equivalent to the statement that the a_j are pairwise relatively prime. □

Given a Seifert manifold M, in order to determine $H^*(M; A)$ and $H_*(M; A)$ it is only necessary to find $H_i(M)$ and $H^i(M)$ for $i = 1, 2$, since for any closed connected orientable 3-manifold, $H_0(M) \approx H^0(M) \approx H_3(M) \approx H^3(M) \approx \mathbb{Z}$. However, once $H_1(M)$ has been determined for any orientable 3-manifold M, $H^1(M) \approx H_2(M)$, $H^2(M)$ can be found by using Poincaré duality and the universal coefficient theorem.

Remark 2. It follows from this observation that $H^2(M) \approx H_1(M)$ and that $H_2(M) \approx H^1(M) \approx \mathrm{hom}(H_1(M); \mathbb{Z}) \approx \mathbb{Z}^t$, where $H_1(M)$ and $t := \mathrm{rank}(H_1(M))$ are given in the Main Theorems.

To state the final result without resorting to a case by case analysis, write: $F_p(H_1(M)) = \mathbb{Z}^t \oplus \mathbb{Z}/p^{r_1} \oplus \cdots \oplus \mathbb{Z}/p^{r_n}$, where, for all cases $t, r_1, \ldots, r_n$ are given by the appropriate Main Theorem.

Corollary 3. $H^2(M; \mathbb{Z}/p^s) \approx H_1(M; \mathbb{Z}/p^s) \approx H_2(M; \mathbb{Z}/p^s) \approx H^1(M; \mathbb{Z}/p^s) \approx (\mathbb{Z}/p^s)^t \oplus \mathbb{Z}/p^{\min(r_1,s)} \oplus \cdots \oplus \mathbb{Z}/p^{\min(r_n,s)}$.

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